Estimate of the number of one-parameter families of modules over a tame algebra

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Abstract

The problem of classifying modules over a tame algebra A reduces to a block matrix problem of tame type whose indecomposable canonical matrices are zero- or one-parameter. Respectively, the set of non-isomorphic indecomposable modules of dimension at most d divides into a finite number $f(d, A)$ of modules and one-parameter series of modules.

We prove that the number of canonical parametric block matrices of size $m \times n$ and a given partition into blocks is bounded by 4^s, where s is the number of free entries, $s \leqslant mn$. Basing on this estimate, we

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prove that

$$f(d,A) \leqslant \binom{d+r}{r} 4^{d^2(\delta_1^2+\cdots+\delta_r^2)} \leqslant (d+1)^r 4^{d^2(\dim A)^2},$$

where r is the number of nonisomorphic indecomposable projective left A-modules and $\delta_1, \ldots, \delta_r$ are their dimensions.

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1 Introduction

Matrices and finite dimensional algebras are considered over an algebraically closed field k.

Gabriel, Nazarova, Roiter, Sergeichuk, and Vossieck [8] studied matrix problems, in which the row-transformations are given by a category and the column transformations are arbitrary. They interpreted $m \times n$ matrices as points of the affine space $k^{m\times n}$ of all $m \times n$ matrices and proved that for a tame matrix problem and every $m \times n$ there exists a full system of nonisomorphic indecomposable $m\times n$ matrices that consists of a finite number of points and punched straight lines. This result was extended to modules over a tame finite dimensional algebra A: for every $d \in \mathbb{N}$ there exists an almost full (except for a finite number of modules) system of nonisomorphic indecomposable d-dimensional modules that consists of a finite number $\rho_A(d)$ of punched lines (an A-module of dimension d was considered as a point of the affine space $k^{d\times d} \oplus \cdots \oplus k^{d\times d}$; the number of summands $k^{d\times d}$ is a number of generators of A).

Brüstle [3] proved, that

$$\rho_A(d) \leqslant \dim(\mathrm{Rad}\, A) \cdot e^{2^6 3^{d-1}(d-1)^{2d-1}}. \tag{1}$$

Sergeichuk [10] extended the results of [8] to block matrix problems in which rows and columns transformations are given by triangular matrix algebras: If the matrix problem is of tame type, then for every $m \times n$ there exists a finite set of zero- and one-parameter matrices

$$M_1, \ldots, M_{t_1},\ N_1(\lambda_1), \ldots, N_{t_2}(\lambda_{t_2}) \tag{2}$$

such that the set of indecomposable canonical $m \times n$ matrices is

$$\{M_1, \ldots, M_{t_1}\} \cup \{N_1(a)\,|\, a \in k\} \cup \cdots \cup \{N_{t_2}(a)\,|\, a \in k\};$$

it may be interpreted as a set of points and straight lines in the affine space $k^{m\times n}$. The proof was based on Belitskiĭ's algorithm [1] (see also [2]) for reducing a matrix to canonical form; two matrices may be reduced one to the other if and only if they have the same canonical form.

Drozd [5] proposed the following reduction of the problem of classifying modules over an algebra A to a matrix problem. Let $P_1, \dots, P_r$ be all nonisomorphic indecomposable projective right A-modules. For every right module M over A, there exists an exact sequence

$$P_1^{p_1} \oplus \cdots \oplus P_r^{p_r} \xrightarrow{\varphi} P_1^{q_1} \oplus \cdots \oplus P_r^{q_r} \xrightarrow{\psi} M \longrightarrow 0,$$

where $X^l := X \oplus \cdots \oplus X$ (l times). The homomorphism φ is determined up to transformations $\varphi \mapsto g\varphi f$, where f and g are automorphisms of $\oplus_i P_i^{p_i}$ and $\oplus_i P_i^{q_i}$. The φ, f, and g can be given by their matrices in bases of the spaces $\oplus_i P_i^{p_i}$ and $\oplus_i P_i^{q_i}$ over k. This reduces the problem of classifying modules over algebras to block matrix problems, which were studied in [10]. The modules that correspond to the canonical matrices form a full system of nonisomorphic modules; indecomposable modules correspond to indecomposable matrices.

In this article, we obtain the following estimates:

(i) If a block matrix problem is of tame type, then the number of canonical parametric block matrices (2) of size $m \times n$ and a given partition into blocks is bounded by 4^s, where s is the number of free entries, $s \leqslant mn$.

(ii) If an algebra A is of tame type, then the number of zero- and one-parameter matrices that give a full system of nonisomorphic indecomposable modules of dimension at most d is bounded by

$$\binom{d+r}{r} 4^{d^2(\delta_1^2+\cdots+\delta_r^2)},$$

where r is the number of nonisomorphic indecomposable projective left A-modules and $\delta_1, \dots, \delta_r$ are their dimensions.

Here the first estimate is optimal and the second one improves significantly the estimate from [3]. The paper is organized as follows: in Section 2, we introduce the concept of standard linear matrix problems and recall Belitskii's algorithm. Section 3 is devoted to the proof of the estimate (i), Section 4 is concerned with the corresponding estimate (ii) for modules over a tame algebra.

2 Belitskiĭ's algorithm for linear matrix problems

A block matrix $M = [M_{ij}]$, $M_{ij} \in k^{m_i \times n_j}$, will be called an $\underline{m} \times \underline{n}$ *matrix*, where $\underline{m} = (m_1, m_2, \dots)$ and $\underline{n} = (n_1, n_2, \dots)$.

A linear matrix problem is the canonical form problem for $\underline{n} \times \underline{n}$ matrices whose blocks satisfy a certain system of linear homogeneous equations. Solving this system, we select *free blocks* that are arbitrary; the other blocks are their linear combinations. The set of admissible transformations consists of elementary transformations within strips, additions of linear combinations of rows of the ith strip to rows of the jth strip for certain $i > j$, and additions of linear combinations of columns of the ith strip to columns of the jth strip for certain $i < j$. Elementary transformations and additions may be linked: making elementary transformations within a horizontal strip, we must produce the same elementary transformations within all horizontal strips linked with it and inverse elementary transformations within all vertical strips linked with it. Making an addition between strips, we must produce all linked with it additions.

Applying Belitskiĭ's algorithm ([1],[10]), we can reduce a block matrix by these transformations to canonical form; two block matrices may be reduced one to the other if and only if they have the same canonical form.

If the matrix problem is of tame type (that is, it does not contain the problem of classifying pairs of matrices up to simultaneous similarity, then the set of direct-sum-indecomposable canonical $\underline{n} \times \underline{n}$ matrices forms a finite number of points and straight lines in the affine space of $\underline{n} \times \underline{n}$ matrices (see [10, Theorem 3]). In the article, we prove that this number is bounded by 4^s, where s is the number of entries in free blocks.

Let us sketch a more formal definition of a linear matrix problem (see [10, Sect. 2.2]).

An algebra $\mathit{\Gamma} \subset k^{t \times t}$ of upper triangular matrices is a *basic matrix algebra* if

$$\begin{bmatrix} a_{11} & \cdots & a_{1t} \\ & \ddots & \vdots \\ 0 & & a_{tt} \end{bmatrix} \in \mathit{\Gamma} \quad \text{implies} \quad \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{tt} \end{bmatrix} \in \mathit{\Gamma}. \tag{3}$$

The diagonals $(a_{11}, a_{22}, \dots, a_{tt})$ of the matrices from $\mathit{\Gamma}$ form a subspace in $k^t = k \oplus \cdots \oplus k$, which may be given by a system of equations of the form

$a_{ii} = a_{jj}$. Define an equivalence relation in $T = \{1, \dots, t\}$ putting

$$i \sim j \text{ if and only if } \operatorname{diag}(a_1, \dots, a_t) \in \Gamma \text{ implies } a_i = a_j. \tag{4}$$

We say that a sequence of nonnegative integers $\underline{n} = (n_1, n_2, \dots, n_t)$ is a *step-sequence* if $i \sim j$ implies $n_i = n_j$.

A *linear matrix problem given by a pair*

$$(\Gamma, \mathcal{M}), \quad \Gamma\mathcal{M} \subset \mathcal{M},\ \mathcal{M}\Gamma \subset \mathcal{M}, \tag{5}$$

consisting of a basic $t \times t$ algebra Γ and a vector space $\mathcal{M} \subset k^{t\times t}$, is the canonical form problem for matrices $M \in \mathcal{M}_{\underline{n}\times\underline{n}}$ with respect to transformations

$$M \longmapsto S^{-1}MS, \qquad S \in \Gamma^*_{\underline{n}\times\underline{n}}, \tag{6}$$

where $\underline{n} = (n_1, \dots, n_t)$ is a step-sequence, $\Gamma_{\underline{n}\times\underline{n}}$ and $\mathcal{M}_{\underline{n}\times\underline{n}}$ consist of $\underline{n} \times \underline{n}$ matrices whose blocks satisfy the same systems of linear homogeneous equations as the entries of $t\times t$ matrices from Γ and $\mathcal{M}$, respectively, and $\Gamma^*_{\underline{n}\times\underline{n}}$ denotes the set of nonsingular matrices from $\Gamma_{\underline{n}\times\underline{n}}$. ($\Gamma$ and $\mathcal{M}$ are subspaces of $k^{t\times t}$; they may be given by systems of linear homogeneous equations of the form

$$\sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} d_{ij}x_{ij} = 0,$$

where $\mathcal{I}, \mathcal{J} \in \{1, \dots, t\}/\!\sim$ are equivalence classes.)

Let us outline Belitskiĭ's algorithm (it has been detailed in [10]) for reducing a matrix

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1t} \\ \dots & \dots & \dots \\ M_{t1} & \cdots & M_{tt} \end{bmatrix} \in \mathcal{M}_{\underline{n}\times\underline{n}}$$

to canonical form by transformations (6). We assume that the blocks of M (and of every block matrix) are ordered starting from the lower strip:

$$M_{t1} < M_{t2} < \cdots < M_{tt} < M_{t-1,1} < M_{t-1,2} < \cdots < M_{t-1,t} < \cdots \tag{7}$$

In the set $\{M_{ij}\}$ of blocks of M, we select the set of free blocks such that every unfree block is a linear combination of free blocks that preceding it with respect to the ordering (7). The entries of free blocks will be called the *free entries.*

On the first step, we reduce the block M_{t1}. It is reduced by transformations

$$M_{t1} \longmapsto S_{tt}^{-1} M_{t1} S_{11}, \qquad S \in \mathit{\Gamma}^*_{\underline{n}\times\underline{n}}. \tag{8}$$

If $1 \nsim t$, then M_{t1} is reduced by arbitrary equivalence transformations. We reduce it to the form

$$\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \tag{9}$$

and extend its division into substrips onto the first vertical and the first horizontal strips of M.

If $1 \sim t$, then M_{t1} is reduced by arbitrary similarity transformations. We reduce it to a *Weyr matrix* (which is obtained from a Jordan matrix by simultaneous permutations of rows and columns, see [10, Sect. 1.3]):

$$W = W_{\alpha_1} \oplus \cdots \oplus W_{\alpha_r}, \quad \alpha_1 \prec \cdots \prec \alpha_r, \tag{10}$$

where $\prec$ is a linear order in k (if k is the field of complex numbers, we use the lexicographic ordering), and

$$W_{\alpha_i} = \begin{bmatrix} \alpha_i I_{m_{i1}} & W_{i1} & & 0 \\ & \alpha_i I_{m_{i2}} & \ddots & \\ & & \ddots & W_{i,q_i-1} \\ 0 & & & \alpha_i I_{m_{iq_i}} \end{bmatrix}, \quad W_{ij} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{11}$$

$m_{i1} \geqslant \ldots \geqslant m_{iq_i}$. We make the most coarse partition of W into substrips for which all diagonal subblocks have the form $\alpha_i I$ and all off-diagonal subblocks are 0 and I (all matrices commuting with W are upper block triangular with respect to this partition). We extend this division of $M_{t1} = W$ into substrips onto the first vertical and the first horizontal strips of M.

Then we restrict the set of admissible transformations with M to those transformations (8) that preserve M_{t1} (that is, $S_{tt}^{-1} M_{t1} S_{11} = M_{t1}$). It may be proved that the algebra of matrices

$$\Lambda_1 = \{ S = [S_{ij}] \in \mathit{\Gamma}_{\underline{n}\times\underline{n}} \,|\, M_{t1} S_{11} = S_{tt} M_{t1} \}$$

also has the form $\mathit{\Gamma}'_{\underline{n'}\times\underline{n'}}$, where $\mathit{\Gamma}'$ is a basic matrix algebra. The entries of M_{t1} are the *reduced entries* of M.

On the second step, we take the first unreduced (that is, does not contained in M_{t1}) block with respect to the new partition and reduce it.

On each step, we take the first unreduced block M_{pq} (with respect to a new subdivision) and reduce it by those admissible transformations that preserve all reduced entries. If M_{pq} is not free, then it is the linear combination of preceding free blocks that have been reduced, and hence M_{pq} is not changed at this step. If M_{pq} is free, then the following three cases are possible:

(i) There exists a nonzero admissible addition to M_{pq} from other blocks. Since admissible transformations are given by upper block triangular matrices and we use the ordering (7), all nonzero additions to M_{pq} are from preceding (reduced) blocks. We make $M_{pq} = 0$ by these additions.

(ii) There exist no nonzero admissible additions to M_{pq} and it is reduced by equivalence transformations. Then we reduce M_{pq} to the form (9).

(iii) There exist no nonzero admissible additions to M_{pq} and it is reduced by similarity transformations. Then we reduce M_{pq} to a Weyr matrix.

At the end of this step, we make an additional subdivision of M into strips in accordance with the block form of the reduced M_{pq} and restrict the set of admissible transformations to those that preserve M_{pq}.

The process stops after reducing the last unreduced entry of M. The obtained canonical matrix will be partitioned into

$$M_1, M_2, \dots, M_{l(M)}, \tag{12}$$

where M_i is the block that reduces at the ith step. Each M_i has the form 0, (8), or is a Weyr matrix. We will call (12) the *boxes* of M.

For instance,

$$M = \left[\begin{array}{c|c|c} \multirow{2}{*}{M_3} & M_6 & M_7 \\ \cline{2-3} & M_4 & M_5 \\ \hline M_1 & \multicolumn{2}{c}{M_2} \end{array}\right] = \left[\begin{array}{cc|c|c} -1 & 1 & 2 & 0 \\ \cline{3-4} 0 & -1 & 0 & 1 \\ \hline \multicolumn{2}{c|}{3I_2} & \multicolumn{2}{c}{0} \end{array}\right], \qquad l(M) = 7,$$

is a canonical $(2,2) \times (2,2)$ matrix for the linear matrix problem given by the pair $(\mathit{\Gamma}, k^{2\times 2})$, where

$$\mathit{\Gamma} = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \,\middle|\, a, b \in k \right\}.$$

Let M be a canonical matrix. Replacing all diagonal entries of its free boxes that are Weyr matrices by parameters, we obtain a parametric matrix $M(\lambda_1, \dots, \lambda_p)$. Its *domain of parameters* $\mathcal{D}$ is the set of all $(a_1, \dots, a_p) \in k^p$ for which $M(a_1, \dots, a_p)$ is a canonical matrix. If a parameter λ_i is finite

(that is, the number of vectors of $\mathcal{D}$ with distinct a_i is finite), we replace λ_i by its values and obtain several parametric matrices with a smaller number of parameters. Repeating this process, we obtain parametric matrices having only infinite parameters. The obtained matrices will be called *canonical parametric matrices.*

Hence, the canonical form problem for $\underline{n} \times \underline{n}$ matrices with the same $\underline{n}$ reduces to the problem of finding a finite number of canonical parametric matrices and their domains of parameters.

3 Estimate of the number of canonical parametric matrices

In this section, we study a linear matrix problem of tame type. As was proved in [10], each of its canonical parametric matrices, up to simultaneous permutations of rows and columns, has the form

$$N_1(\lambda_1) \oplus \cdots \oplus N_p(\lambda_p) \oplus R_1 \oplus \cdots \oplus R_q, \qquad p \geqslant 0, \quad q \geqslant 0, \tag{13}$$

where $N_i(\lambda_i)$ and R_j are indecomposable canonical one- and zero-parameter canonical matrices. The purpose of the section is to prove the following theorem.

Theorem 3.1. *If a linear matrix problem is of tame type, then the number of its canonical parametric matrices of size $\underline{n} \times \underline{n}$ is bounded by $4^{s(\underline{n})}$, where $s(\underline{n})$ is the number of free entries in an $\underline{n} \times \underline{n}$ matrix.*

We first prove a technical lemma.

Lemma 3.1. *Let*

$$A(x,y) = \begin{bmatrix} a_{11}(x,y) & \dots & a_{1n}(x,y) \\ \dots & \dots & \dots \\ a_{m1}(x,y) & \dots & a_{mn}(x,y) \end{bmatrix} \tag{14}$$

be a matrix whose entries are linear polynomials in x and y, and let the rows of $A(\alpha,\beta)$ be linearly independent for all $(\alpha,\beta) \in k^2$ except for

$$(\alpha_1,\beta_1),\ (\alpha_2,\beta_2),\dots,(\alpha_s,\beta_s).$$

Then $s \leqslant m^2$; moreover, $s \leqslant 3$ if $m = 2$.

Proof. Part 1: $s \leqslant m^2$. Clearly, $m \leqslant n$. The rows of $A(\alpha,\beta)$ are linearly dependent if and only if $(\alpha,\beta) \in k^2$ is a common root of all determinants formed by columns of $A(x,y)$. The determinants are polynomials in x and y of degree at most m; they are relatively prime (otherwise, they have infinitely many common roots $(\alpha,\beta) \in k^2$). The inequality $s \leqslant m^2$ follows from the following statement:

$$\begin{array}{l}\text{If } h_1,\dots,h_t \in k[x,y] \text{ are polynomials of degree at most } m \\ \text{and their greatest common divisor } (h_1,\dots,h_t) \text{ is 1, then} \\ \text{they have at most } m^2 \text{ common roots.}\end{array} \tag{15}$$

For $m = 2$, this statement is a partial case of the Bezout theorem [9, Sect. 1.3]: if $h_1, h_2 \in k[x,y]$ and $(h_1,h_2) = 1$, then they have at most $\deg(h_1)\cdot\deg(h_2)$ common roots.

Let $m \geqslant 3$. Applying induction in t, we may assume that $d := (h_1,\dots,h_{t-1}) \neq 1$. If (α,β) is a common root of $h_1,\dots,h_t$, then (α,β) is a root of h_t and also a root of d or a common root of $g_1 = h_1/d,\dots,g_{t-1} = h_{t-1}/d$. By the Bezout theorem, the number of common roots of d and h_t is at most $\deg(d)m$. By induction, the number of common roots of $g_1,\dots,g_{t-1}$ is at most $(m-\deg(d))^2$. Hence, the number of common roots of $h_1,\dots,h_t$ is at most $\deg(d)m+(m-\deg(d))^2 \leqslant \deg(d)m+(m-\deg(d))m = m^2$. This proves (15).

Part 2: $s \leqslant 3$ if $m = 2$. Let $m = 2$; assume to the contrary that $s > 3$. We will reduce $A(x,y)$ by elementary transformations over k and by substitutions

$$\begin{aligned} x_{\text{new}} &= ax+by+c, \\ y_{\text{new}} &= a_1x+b_1y+c_1, \end{aligned} \qquad \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \neq 0;$$

the obtained matrices $A'(x,y)$ will have the same number s, and their entries are linear polynomials too. We suppose that each of the matrices $A'(x,y)$ does not contain a zero column; otherwise we can remove it and take the obtained matrix instead of $A(x,y)$.

Let $n = 2$. The rows of $A(\alpha,\beta)$ are linearly independent only if $\det A(\alpha,\beta) \neq 0$. Under the conditions of the lemma, the rows of $A(\alpha,\beta)$ are linearly independent for almost all $(\alpha,\beta) \in k^2$, and so $\det A(x,y)$ is a nonzero scalar and the rows of $A(\alpha,\beta)$ are linearly independent for all $(\alpha,\beta) \in k^2$.

Hence, $n \geqslant 3$. By elementary transformations of rows of $A(x,y)$, we make $a_{11}(x,y) = a_{11} \in \{0,1\}$.

If $a_{21}(x,y) = a_{21} \in k$, we make $(a_{11}, a_{21}) = (1, 0)$ by elementary transformations of rows. The rows of $A(\alpha, \beta)$ are linearly dependent only if

$$a_{22}(\alpha,\beta) = a_{23}(\alpha,\beta) = \cdots = a_{2n}(\alpha,\beta) = 0.$$

Since $a_{22}(x,y), a_{23}(x,y), \dots$ are linear polynomial, $s \leqslant 1$.

Hence $a_{21}(x,y) \notin k$. We make $a_{21}(x,y) = x$ by the substitution

$$x_{\text{new}} = a_{21}(x,y), \quad y_{\text{new}} = \begin{cases} y & \text{if } a_{21}(x,y) \notin k[y], \\ x & \text{otherwise.} \end{cases}$$

If there exist distinct $l, r > 1$ such that

$$\begin{aligned} a_{1l}(x,y) &= ax + by + c, \\ a_{1r}(x,y) &= a_1 x + b_1 y + c_1, \end{aligned} \qquad \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \neq 0, \tag{16}$$

then we make $a_{12}(x,y) = x + a$ by elementary transformations of columns except for the first column. The rows of $A(\alpha, \beta)$ are linearly dependent if and only if (α, β) is a solution of the system

$$\begin{vmatrix} a_{11}(x,y) & a_{1j}(x,y) \\ a_{21}(x,y) & a_{2j}(x,y) \end{vmatrix} = \begin{vmatrix} a_{11} & a_{1j}(x,y) \\ x & a_{2j}(x,y) \end{vmatrix} = 0, \quad j = 2, \dots, m. \tag{17}$$

The first equation has the form

$$\begin{vmatrix} a_{11} & x + a \\ x & bx + cy + d \end{vmatrix} = 0. \tag{18}$$

Let $a_{11} c \neq 0$. We present (18) in the form $y = a_1 x^2 + b_1 x + c_1$, substitute it into the other equations of the system (17), and obtain a system of polynomial equations in x of degree at most 3. This system has at most three solutions, and so $s \leqslant 3$.

Let $a_{11} c = 0$. Since (18) is a quadratic equation in x, $x = \alpha_1$ or $x = \alpha_2$ for certain $\alpha_1, \alpha_2 \in k$. Substituting $x = \alpha_i$ into the other equations of the system (17) gives a system of linear equations with respect to y, which has at most one solution, and so $s \leqslant 2$.

Hence, (16) does not hold for all $l, r > 1$. If there exists $j > 1$ such that $a_{1j}(x,y) = bx + a$, $b \neq 0$, then we make $b = 1$ and reason as in the previous case. The case $a_{1j}(x,y) = a_j \in k$ for all $j > 1$ is trivial. Let us consider the remaining case $a_{1j}(x,y) = ax + by + c$, $b \neq 0$, for a certain $j > 1$. We make

$$A(x,y) = \begin{bmatrix} a_{11} & y & 0 & \dots & 0 \\ x & a_{22}(x,y) & a_{23}(x,y) & \dots & a_{2n}(x,y) \end{bmatrix}$$

by the substitution $y_{\text{new}} = ax + by + c$ and by elementary transformations of columns starting with the second. If $a_{11} = 0$, then the rows of $A(\alpha, 0)$ are linearly dependent for all $\alpha \in k$. Hence $a_{11} = 1$.

If the system

$$a_{2j}(x, y) = 0, \quad j = 3, \dots, n,$$

has at most one solution, then $s \leqslant 1$. So this system is equivalent to one equation of the form $y = ax + b$ or $x = a$. Substituting it into

$$\begin{vmatrix} a_{11} & y \\ x & a_{22}(x, y) \end{vmatrix} = 0,$$

we obtain a quadratic equation with respect to x or y. Hence $s \leqslant 2$, a contradiction. $\square$

Let a linear matrix problem of tame type be given by a pair $(\Gamma, \mathcal{M})$ and let $M \in \mathcal{M}_{\underline{n} \times \underline{n}}$. We sequentially reduce M to the canonical parametric form. If a block is reduced to a Weyr matrix, we replace its diagonal entries by parameters; but as soon as it becomes clear from the form of subsequent boxes in the process of reduction that a parameter may possess only a finite number of values, we replace it by these values.

The matrix that is obtained after reduction of the first r boxes will be called an *r-matrix*; its partition into strips (which refines the $\underline{n} \times \underline{n}$ partition) will be called the *r-partition*, its strips and blocks will be called *r-strips* and *r-blocks*. Two r-matrices are *equivalent* if their reduced boxes coincide.

Let M be an r-matrix. Denote by $\bar{M}$ the matrix obtained from it by replacement of all unreduced free entries with zeros. Since the matrix problem is of tame type, $\bar{M}$ is canonical for all values of parameters, and it is reduced by simultaneous permutations of horizontal and vertical r-strips to the form

$$\bar{M}^{\vee} = N_1(\lambda_1 I) \oplus \cdots \oplus N_p(\lambda_p I) \oplus (R_1 \otimes I) \oplus \cdots \oplus (R_q \otimes I), \tag{19}$$

where $N_i(\lambda_i I)$ and $R_j \otimes I$ are indecomposable canonical one- and zero-parameter canonical matrices ($R_j \otimes I$ is obtained from R_j by replacement of all its entries a with aI).

By the same permutation of r-strips, we reduce M to $M^{\vee}$ and break up it into $(p+q) \times (p+q)$ strips conformally to (19). The obtained strips and blocks will be called the *big strips* and *big blocks* of $M^{\vee}$. (In the terminology of [10], the r-strips of M that are contained in the same big strip are *linked*.)

Define the *weight*

$$t_M = 3^{w(M)}$$

of an r-matrix M, where $w(M)$ is the number of entries in all free boxes M_i, $i \leqslant r$, with the following property: M_i disposes in the same big strip with a free box M_L, $L < i$, containing a parameter (that is, M_i is linked with a box having a parameter and reduces after it). Denote by $s(M)$ the number of free entries in the first unreduced r-block of M.

We say that an $(r+1)$-canonical matrix B is an *extension* of an r-canonical matrix M and write $B \supset M$ if the boxes $B_1, B_2, \dots, B_r$ coincide with the boxes $M_1, M_2, \dots, M_r$ or are obtained from them by replacement of some of their parameters by scalars.

The proof of Theorem 3.1 bases on the following lemma.

Lemma 3.2. *Let M be an r-matrix having unreduced entries. Then the number of its nonequivalent extensions $B \supset M$ taken t_B/t_M times is at most $4^{s(M)}$:*

$$\sum_{\text{nonequiv. } B \supset M} t_B/t_M \leqslant 4^{s(M)}. \tag{20}$$

Proof. Let M_{r+1} be the first unreduced r-block of M and let $M^{\vee}_{xy}$ be the big block containing M_{r+1}. The following three cases are possible.

Case 1: $x > p$ and $y > p$ (see (19)). Then the horizontal and the vertical big strips of $M^{\vee}_{xy}$ do not contain parameters, and $t_B = t_M$ for all $B \supset M$.

(i) Let there exist a nonzero addition to M_{r+1}. We make $M_{r+1} = 0$, then all $B \supset M$ are equivalent and the inequality (20) takes the form $1 \leqslant 4^{s(M)}$.

(ii) Let there exist no nonzero addition to M_{r+1} and M_{r+1} is reduced by elementary transformations. Then each $B \supset M$ has B_{r+1} of the form (9), the number of such $z_1 \times z_2$ matrices B_{r+1} is $\min\{z_1, z_2\} + 1$. The inequality (20) takes the form $\min\{z_1, z_2\} + 1 \leqslant 4^{z_1 z_2}$.

(iii) Let there exist no nonzero addition to M_{r+1} and M_{r+1} is reduced by similarity transformations. Then the box B_{r+1} of each $B \supset M$ is a parametric Weyr matrix. The number of parametric $z \times z$ Weyr matrices is bounded by 3^{z-1} since the structure of a matrix W of the form (10) is determined by the sequence $(n_2, \dots, n_z) \in \{1, 2, 3\}^{z-1}$, where $n_l = 1$ if the (l, l) entry of W is the first entry of W_{α_i}, $n_l = 2$ if the (l, l) entry is not the first entry of W_{α_i} but the first entry of $\alpha_i I_{m_{ij}}$ (see (11)), and $n_l = 3$ if the (l, l) entry is not the first entry of $\alpha_i I_{m_{ij}}$. Hence, the number of nonequivalent extensions B of M is bounded by 3^{z-1}. This proves (20) since $t_B = t_M$ and $s(M) = z^2$.

Case 2: $x \leqslant p < y$ or $y \leqslant p < x$. Then a horizontal or vertical big strip of $M^{\vee}_{xy}$ contains a parameter λ_l, $l \in \{1, \dots, p\}$.

Let the parameters of M take on values from the domain of parameters. There exists no nonzero addition to M_{r+1} if and only if

$$M' = SMS^{-1} \tag{21}$$

implies $M'_{r+1} = M_{r+1}$ for all r-matrices M' that are equivalent to M and all $S \in \Gamma_{\underline{n}\times\underline{n}}$ whose main diagonal with respect to r-partition consists of the identity r-blocks.[1]

Let us partition S and M into r-blocks: $S = [S_{\alpha\beta}]^e_{\alpha,\beta=1}$ and $M = [M_{\alpha\beta}]^e_{\alpha,\beta=1}$. Since M_{r+1} is an r-block, $M_{r+1} = M_{\zeta\eta}$ for certain ζ and η. Presenting (21) in the form $M'S = SM$ and equating the (ζ, η) r-blocks, we obtain

$$M'_{\zeta 1}S_{1\eta} + \cdots + M'_{\zeta,\eta-1}S_{\eta-1,\eta} + M'_{\zeta\eta} = M_{\zeta\eta} + S_{\zeta,\zeta+1}M_{\zeta+1,\eta} + \cdots + S_{\zeta e}M_{e\eta} \tag{22}$$

since S is upper triangular with identity diagonal r-blocks.

The blocks $M'_{\zeta 1}, \dots, M'_{\zeta,\eta-1}$ precede $M'_{\zeta\eta}$ so they have been reduced and $M'_{\zeta 1} = M_{\zeta 1}, \dots, M'_{\zeta,\eta-1} = M_{\zeta,\eta-1}$. Moreover, each of them is nonzero only when it is contained in the big block $M^{\vee}_{xx}$ (they are contained in the x big horizontal strip of $M^{\vee}$ since $M_{\zeta\eta}$ is contained in $M^{\vee}_{xy}$, but $M^{\vee}$ is big-block-diagonal, see (19)). Analogously, each of $M_{\zeta+1,\eta}, \dots, M_{e\eta}$ is nonzero only when it is contained in $M^{\vee}_{yy}$. Hence, each r-block $S_{\alpha\beta}$ in (22) may have a nonzero factor only when it is contained in $S^{\vee}_{xy}$. This factor has the form $(a\lambda_l + b)I$, $a, b \in k$, since all reduced free r-blocks from $M^{\vee}_{xx}$ and $M^{\vee}_{yy}$ are zero matrices, scalar matrices, and $\lambda_l I$.

Therefore, there exists no nonzero addition to M_{r+1} for $\lambda_l = a \in k$ if and only if the following property holds for each $S \in \Gamma_{\underline{n}\times\underline{n}}$ whose main diagonal with respect to r-partition consists of the identity r-blocks: if the transformation (21) given by S preserves all boxes preceding M_{r+1}, then

$$M_{\zeta 1}S_{1\eta} + \cdots + M_{\zeta,\eta-1}S_{\eta-1,\eta} - S_{\zeta,\zeta+1}M_{\zeta+1,\eta} - \cdots - S_{\zeta e}M_{e\eta} = 0. \tag{23}$$

The equality (23) is a linear combination of r-blocks from $S^{\vee}_{xy}$; its coefficients are linear polynomials in λ_l.

[1] In [10, Theorem 1.4(b)], the condition "but $M'_q \neq M_q$" must be replaced with "and $M'_q = 0$".

The conditions on r-blocks of $S^{\vee}_{xy}$ that ensure the preservation of all boxes preceding M_{r+1} can be formulated in the form of a system of linear homogeneous equations with respect to r-blocks of S that consists of:

(a) Linear equations with coefficients from k that give the algebra $\Gamma_{\underline{n}\times\underline{n}}$ as a vector space. We restrict ourselves to those equations that contain r-blocks from $S^{\vee}_{xy}$, then they do not contain r-blocks outside $S^{\vee}_{xy}$ (see [10, p. 87]).

(b) Linear equations with coefficients from k that ensure the preservation of those free r-blocks $M_{\alpha\beta}$ that are contained in the intersection of $M^{\vee}_{xy}$ with the boxes $M_1, \dots, M_L$, where M_L is the free box containing the parameter λ_l. These equations have the form (23) with the indices (α, β) instead of (ζ, η).

(c) Linear equations, whose coefficients are linear polynomials in λ_l, that ensure the preservation of free r-blocks $M_{\alpha\beta}$ contained in the intersection of $M^{\vee}_{xy}$ with the boxes $M_{L+1}, \dots, M_r$; the number of entries in the boxes $M_{\alpha\beta}$ will be denoted by h. They also have the form (23) with (α, β) instead of (ζ, η).

Solving the system (a)$\cup$(b), we choose r-blocks $S_1, \dots, S_n$ from $S^{\vee}_{xy}$ such that they are arbitrary and the other r-blocks from $S^{\vee}_{xy}$ are their linear combinations. Substituting the solution into the system (c) and the equation (23), we obtain a system of the form

$$\begin{aligned} a_{11}(\lambda_l)S_1 + \cdots + a_{1n}(\lambda_l)S_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m-1,1}(\lambda_l)S_1 + \cdots + a_{m-1,n}(\lambda_l)S_n = 0 \end{aligned} \tag{24}$$

and, respectively, an equation

$$a_{m1}(\lambda_l)S_1 + \cdots + a_{mn}(\lambda_l)S_n = 0, \tag{25}$$

where $a_{ij}(\lambda_l)$ are linear polynomials in λ_l. We take the equations (24)–(25) such that the $m \times n$ matrix $A(\lambda_l) = [a_{ij}(\lambda_l)]$ has linearly independent rows for almost all values of λ_l; it is possible by [10, Sect. 3.3.2] since the matrix problem is of tame type. Then $m \leqslant n$.

Let there exist no nonzero addition to M_{r+1} for $\lambda_l = \alpha \in k$. Then the equation (25) follows from the system (24). Therefore, all determinants formed by columns of the matrix $A(\lambda_l)$ become zero for $\lambda_l = \alpha$. These determinants are polynomials in λ_l of degree at most m. If all the polynomials are identically equal to 0, then the rows of $A(\lambda_l)$ are linearly dependent for all values of λ_l and the problem is of wild type. Therefore, they have at most m common roots, and hence there are at most m values $\alpha \in k$ of λ_l for which we cannot make $M_{r+1} = 0$.

Let λ_l be equal to one of these values. The matrix M_{r+1} is transformed by equivalence transformations since M_{r+1} is not contained in a diagonal big block. Hence each extension $B \supset M$ has B_{r+1} in the form (9); the number of nonequivalent extensions B with nonzero B_{r+1} and the same value of λ_l is $\min\{z_1, z_2\}$, where $z_1 \times z_2$ is the size of M_{r+1}; their weight $t_B \leqslant t_M/3^{m-1}$ (since λ_l no longer is a parameter and $m-1 \leqslant h$, where h is defined in paragraph (c)).

There is also one (up to equivalence) extension $B \supset M$ with $B_{r+1} = 0$ and the parameter λ_l. Its weight $t_B = t_M \cdot 3^{z_1 z_2}$.

We have

$$\sum_{\text{nonequiv. } B\supset M} t_B/t_M \leqslant 3^{z_1 z_2} + m \cdot \min\{z_1, z_2\} \cdot 3^{-m+1} \leqslant 4^{z_1 z_2} = 4^{s(M)}$$

since $m \cdot 3^{-m+1} \leqslant 1$ and $3^{z_1 z_2} + \min\{z_1, z_2\} \leqslant 4^{z_1 z_2}$ for all natural numbers m, z_1 and z_2. This proves (20).

Case 3: $x \leqslant p$ and $y \leqslant p$. Then the horizontal and vertical big strips of $M^{\vee}_{xy}$ contain parameters λ_l and λ_r from free boxes M_L and M_R, respectively. We will assume $L \leqslant R$.

Let $l = r$. Then $M_L = M_R$ is a Weyr matrix, $\lambda_l = \lambda_r$ is the parameter of its block (11), and $x = y$. This case is similar to Case 2, but the matrix M_{r+1} is reduced by similarity transformations since M_{r+1} is contained in the diagonal big block $M^{\vee}_{xx}$. In each extension $B \supset M$, the box B_{r+1} is a Weyr matrix. The number of parametric $z \times z$ Weyr matrices is bounded by 3^{z-1} (see Case 1(iii)), so we have

$$\sum_{\text{nonequiv. } B\supset M} t_B/t_M \leqslant 3^{z^2} + m \cdot 3^{z-1} \cdot 3^{-m+1} \leqslant 4^{z^2} = 4^{s(M)}$$

since $m \cdot 3^{-m+1} \leqslant 1$ and $3^{z^2} + 3^{z-1} \leqslant 4^{z^2}$ for all natural numbers m and z.

Let $l \neq r$. Then $x \neq y$; in distinction to Case 2, the system (c) consists of linear equations whose coefficients are linear polynomials in λ_l and λ_r. Correspondingly, the system (24) and the equation (25) take the form

$$\begin{array}{c} a_{11}(\lambda_l, \lambda_r)S_1 + \cdots + a_{1n}(\lambda_l, \lambda_r)S_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m-1,1}(\lambda_l, \lambda_r)S_1 + \cdots + a_{m-1,n}(\lambda_l, \lambda_r)S_n = 0 \end{array} \tag{26}$$

and

$$a_{m1}(\lambda_l, \lambda_r)S_1 + \cdots + a_{mn}(\lambda_l, \lambda_r)S_n = 0, \tag{27}$$

respectively, where $a_{ij}(\lambda_l,\lambda_r)$ are linear polynomials in λ_l and λ_r.

Let there exist no nonzero addition to M_{r+1} for $(\lambda_l,\lambda_r)=(\alpha,\beta)\in k^2$. Then the equation (27) follows from the system (26) and hence the matrix $A(\alpha,\beta)$ (see (14)) has linearly dependent rows. The set of values of (λ_l,λ_r) for which the rows of $A(\lambda_l,\lambda_r)$ are linearly dependent is finite (otherwise the matrix problem is of wild type, see [10, Sect. 3.3.1]); assume that this set consists of pairs (α_1,β_1), $(\alpha_2,\beta_2),\dots,(\alpha_s,\beta_s)\in k^2$.

By analogy with Case 2, there are at most $s\cdot\min\{z_1,z_2\}$ nonequivalent extensions $B\supset M$ with nonzero B_{r+1} of size $z_1\times z_2$, their weight $t_B\leqslant t_M/3^{m-1}$ (since λ_l and λ_r no longer are parameters). There is also one extension $B\supset M$ with $B_{r+1}=0$ and the parameters λ_l and λ_r; its weight $t_B=t_M\cdot 3^{z_1z_2}$. We have

$$\sum_{\text{nonequiv. } B\supset M} t_B/t_M\leqslant 3^{z_1z_2}+s\cdot\min\{z_1,z_2\}\cdot 3^{-m+1}\leqslant 4^{z_1z_2}=4^{s(M)}$$

since $s\cdot 3^{-m+1}\leqslant 1$ by Lemma 3.1 and $3^{z_1z_2}+\min\{z_1,z_2\}\leqslant 4^{z_1z_2}$ for all natural numbers m, z_1 and z_2. This proves (20). $\square$

Proof of Theorem 3.1. Let M be an r-matrix of size $\underline{n}\times\underline{n}$. We will write $M\Subset C$ if C is a canonical parametric matrix whose boxes $C_1,C_2,\dots,C_r$ coincide with the boxes $M_1,M_2,\dots,M_r$ or are obtained from them by replacement of some of their parameters by scalars. We may add sequentially the boxes of C to the boxes of M and obtain a sequence of extensions

$$M\subset B_1\subset B_2\subset\cdots\subset B_{l-1}\subset B_l=C, \tag{28}$$

where B_i is an $(r+i)$-matrix and $l+r$ is the number of boxes of C. The length l of this sequence may be changed if we change C; the greatest length l will be called the *dept* of M and will be denoted by $l(M)$.

We prove by induction in $l(M)$ that

$$\sum_{C\Supset M} t_C/t_M\leqslant 4^{\bar{s}(M)}, \tag{29}$$

where $\bar{s}(M)$ is the number of unreduced free entries in M.

If $l(M)=1$, this inequality follows from Lemma 3.2. Let $l(M)\geqslant 2$ and

(29) holds for all r'-matrices whose dept is less than $l(M)$. Then

$$\begin{aligned}
\sum_{C \Supset M} t_C/t_M &= \sum_{\text{nonequiv. } B \supset M} \sum_{C \Supset B} t_C/t_B \cdot t_B/t_M \\
&= \sum_{\text{nonequiv. } B \supset M} t_B/t_M \sum_{C \Supset B} t_C/t_B \\
&\leqslant \sum_{\text{nonequiv. } B \supset M} t_B/t_M \cdot 4^{\bar{s}(B)} && \text{by the induction hypothesis} \\
&= 4^{\bar{s}(M)-s(M)} \sum_{\text{nonequiv. } B \supset M} t_B/t_M \\
&= 4^{\bar{s}(M)-s(M)} \cdot 4^{s(M)} && \text{by Lemma 3.2} \\
&= 4^{\bar{s}(M)};
\end{aligned}$$

that proves (29). The substitution of the 0-canonical matrix 0 for M in (29) gives

$$\sum_{C \Supset 0} t_C \leqslant 4^{s(\underline{n})}.$$

This proves Theorem 3.1 since the sum is taken over all canonical parametric matrices and $t_C \geqslant 1$ by the definition of weight. $\square$

Now we extend Theorem 3.1 to matrix problems, in which row- and column-transformations are separated.

Let $\mathit{\Gamma} \subset k^{t\times t}$ and $\Delta \subset k^{l\times l}$ be two basic matrix algebras and let $\mathcal{N} \subset k^{t\times l}$ be a vector space such that

$$\mathit{\Gamma}\mathcal{N} \subset \mathcal{N} \quad \text{and} \quad \mathcal{N}\Delta \subset \mathcal{N}.$$

By a *separated matrix problem given by* $(\mathit{\Gamma}, \Delta, \mathcal{N})$, we mean the canonical form problem for matrices $N \in \mathcal{N}_{\underline{m}\times\underline{n}}$ in which the row transformations are given by $\mathit{\Gamma}$ and the column transformations are given by Δ:

$$N \longmapsto CNS, \quad C \in \mathit{\Gamma}^*_{\underline{m}\times\underline{m}},\ S \in \Delta^*_{\underline{n}\times\underline{n}}.$$

Following [10, Lemma 2.3], we may consider this matrix problem as the linear matrix problem given by the pair $(\mathit{\Gamma} \times \Delta,\ 0\diagdown\mathcal{N})$ (see (5)), where $0\diagdown\mathcal{N}$ denotes the vector space of $(t+l)\times(t+l)$ matrices of the form

$$\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}, \qquad X \in \mathcal{N}.$$

This permits to extend Theorem 3.1 to separated matrix problems.

Theorem 3.2. *If a separated matrix problem is of tame type, then the number of its canonical parametric matrices of size* $\underline{m}\times\underline{n}$ *is bounded by* $4^{s(\underline{m},\underline{n})}$*, where* $s(\underline{m},\underline{n})$ *is the number of free entries in an* $\underline{m}\times\underline{n}$ *matrix.*

4 Number of modules

The problem of classifying modules over finite dimensional algebra A reduces to a linear matrix problem; its canonical matrices determine a full system of nonisomorphic modules over A (see [10, Sect. 2.5]), which will be called *canonical*. If A is of tame type, then the set of canonical right modules of a fixed dimension partitions into a finite number of series that are determined by canonical parametric matrices of the form (13). In this section, we prove the following estimate.

Theorem 4.1. *If* A *is an algebra of tame type and* $f(d, A)$ *is the number of series of canonical right* A*-modules of dimension at most* d*, then*

$$f(d,A) \leqslant \binom{d+r}{r} 4^{d^2(\delta_1^2+\cdots+\delta_r^2)} \leqslant (d+1)^r 4^{d^2(\dim A)^2}, \tag{30}$$

where r *is the number of nonisomorphic indecomposable projective left* A*-modules, and* $\delta_1,\ldots,\delta_r$ *are their dimensions.*

Without loss of generality, we will prove Theorem 4.1 for basic matrix algebras (see (3)). Indeed, A is isomorphic to the subalgebra $B \subset \operatorname{End}_k A$ consisting of all linear operators

$$\hat{a} : x \mapsto ax, \qquad a \in A, \tag{31}$$

on the space ${}_kA$. There exists a basis of ${}_kA$ in which the matrices of B form an algebra $\Gamma_{\underline{n}\times\underline{n}}$, where $\Gamma \subset k^{t\times t}$ is a basic matrix algebra and $\underline{n} = (n_1,\ldots,n_t) \in \mathbb{N}^t$, see [10, Theorem 1.1]. By the Morita theorem [7], the categories of representations of $\Gamma_{\underline{n}\times\underline{n}}$ and its basic algebra Γ are equivalent, hence

$$f(d,A) = f(d,\Gamma_{\underline{n}\times\underline{n}}) = f(d,\Gamma).$$

Furthermore, the replacement of $\Gamma_{\underline{n}\times\underline{n}}$ with Γ preserves the number r of nonisomorphic indecomposable projective left modules and reduces their dimensions.

The algebra Γ determines the equivalence relation (4) in the set of indices $T = \{1, \dots, t\}$. Let $\mathcal{I}_1, \dots, \mathcal{I}_r$ be the equivalence classes, put

$$e_\alpha = \sum_{i \in \mathcal{I}_\alpha} e_{ii}, \tag{32}$$

where e_{ij} are the matrix units of $k^{t\times t}$. Define the matrix

$$L = [l_{\alpha\beta}]_{\alpha,\beta=1}^{r}, \qquad l_{\alpha\beta} = \dim e_\alpha R e_\beta, \tag{33}$$

where $R = \operatorname{Rad} \Gamma$ is the radical of Γ consisting of all its matrices with zero diagonal.

Lemma 4.1. *If $\Gamma \in k^{t\times t}$ is a basic matrix algebra of tame type, then*

$$f(d, \Gamma) \leqslant \sum_{q_1+\dots+q_r \leqslant d} 4^{[q_1,\dots,q_r]L\cdot([q_1,\dots,q_r]L)^T}, \tag{34}$$

where $q_1, \dots, q_r$ are nonnegative integers.

Let us show that (34) implies Theorem 4.1. By (32),

$$I = e_1 + \dots + e_r$$

is a decomposition of the identity of Γ into a sum of minimal orthogonal idempotents, and so $\Gamma e_1, \dots, \Gamma e_r$ are all nonisomorphic indecomposable projective left modules over Γ. The number of summands in (34) is equal to the number of solutions of the inequality

$$x_1 + \dots + x_r \leqslant d \tag{35}$$

in nonnegative integers; it equals $\binom{d+r}{r}$ by [11, Sect. 1.2]. Since $q_\alpha \leqslant d$, $[q_1, \dots, q_r]L\cdot([q_1, \dots, q_r]L)^T \leqslant d^2[1, \dots, 1]L\cdot([1, \dots, 1]L)^T = d^2(\delta_1^2+\dots+\delta_r^2)$, where $\delta_\beta = [1, \dots, 1] \cdot [l_{1\beta}, \dots, l_{r\beta}]^T = l_{1\beta} + \dots + l_{r\beta} = \dim e_1 R e_\beta + \dots + \dim e_r R e_\beta = \dim(e_1 + \dots + e_r)R e_\beta = \dim R e_\beta = \dim \Gamma e_\beta - 1$. This proves the first inequality in (30). We have

$$\binom{d+r}{r} \leqslant (d+1)^2$$

since each x_i in (35) possesses at most $d+1$ values $0, 1, \dots, d$. We also have $\delta_1^2 + \dots + \delta_r^2 \leqslant (\delta_1 + \dots + \delta_r)^2 = (\dim \Gamma e_1 + \dots + \dim \Gamma e_r)^2 = (\dim \Gamma(e_1 + \dots + e_r))^2 = (\dim \Gamma)^2 \leqslant (\dim A)^2$. This proves the second inequality in (30).

Proof of Lemma 4.1. Step 1: reduction to a matrix problem. The reduction to a linear matrix problem given in [10] is a light modification of Drozd's reduction [5] (see also [6] and [4]). It bases on the construction, for every right module M over Γ, an exact sequence

$$P \xrightarrow{\varphi} Q \xrightarrow{\psi} M \longrightarrow 0, \tag{36}$$

$$\operatorname{Ker}\varphi \subset \operatorname{Rad} P, \quad \operatorname{Im}\varphi \subset \operatorname{Rad} Q, \tag{37}$$

where P and Q are projective right modules. The homomorphism φ is defined by P, Q, and M up to transformations

$$\varphi \longmapsto g\varphi f, \qquad f \in \operatorname{Aut}_\Gamma P, \quad g \in \operatorname{Aut}_\Gamma Q. \tag{38}$$

Let us show briefly (details in [10]) that the problem of classifying φ up to these transformations reduces to a separated matrix problem given by the triple $(\Gamma, \Gamma, \operatorname{Rad}\Gamma)$.

Decompose P and Q from (36) into direct sums of indecomposable projective modules:

$$P = (e_1\Gamma)^{p_1} \oplus \cdots \oplus (e_r\Gamma)^{p_r}, \quad Q = (e_1\Gamma)^{q_1} \oplus \cdots \oplus (e_r\Gamma)^{q_r}, \tag{39}$$

where $X^l := X \oplus \cdots \oplus X$ (l times) and e_i are defined by (32). Then the homomorphism φ becomes the $q \times p = (q_1 + \cdots + q_r) \times (p_1 + \cdots + p_r)$ matrix $\varphi = [\varphi_{xy}]_{x=1,}^{q}{}_{y=1}^{p}$, which we partition into r horizontal and r vertical strips of sizes $q_1, \ldots, q_r$ and $p_1, \ldots, p_r$. Denote by

$$\alpha = \alpha(x) \quad \text{and} \quad \beta = \beta(y)$$

the indices of the vertical and the horizontal strips containing φ_{xy}. Then $\varphi_{xy} : e_\beta\Gamma \to e_\alpha\Gamma$ and is determined by $\varphi_{xy}(e_\beta) = e_\alpha\varphi_{xy}(e_\beta) \in e_\alpha\Gamma$. Since φ_{xy} is a homomorphism and e_β is an idempotent, $\varphi_{xy}(e_\beta) = \varphi_{xy}(e_\beta^2) = \varphi_{xy}(e_\beta)e_\beta$. Hence, $\varphi_{xy}(e_\beta) = e_\alpha\varphi_{xy}(e_\beta)e_\beta \in e_\alpha\Gamma e_\beta$. By (37),

$$\operatorname{Im}\varphi \subset \operatorname{Rad} Q = (e_1R)^{q_1} \oplus \cdots \oplus (e_rR)^{q_r},$$

where $R = \operatorname{Rad}\Gamma$. We have $\varphi_{xy}(e_{\beta(y)}) \in e_{\alpha(x)}Re_{\beta(y)}$.

If a matrix $a = [a_{ij}]_{i,j=1}^t \in \Gamma$ belongs to $e_\alpha Re_\beta$, then it is determined by its submatrix $\bar{a} = [a_{ij}]_{(i,j)\in\mathcal{I}_\alpha\times\mathcal{I}_\beta}$ since all entries outside of $\bar{a}$ are zero by (32). The size of $\bar{a}$ is $h(\alpha) \times h(\beta)$, where $h(\alpha)$ is the number of elements in $\mathcal{I}_\alpha$.

Therefore, the homomorphism $\varphi=[\varphi_{xy}]_{x=1,}^{q}{}_{y=1}^{p}$ is determined by the block matrix

$$[\overline{\varphi_{xy}(e_{\beta(y)})}]_{x=1,}^{q}{}_{y=1}^{p} \tag{40}$$

of size

$$(q_1h(1)+\dots+q_rh(r))\times(p_1h(1)+\dots+p_rh(r)).$$

Permuting rows and columns of this matrix to order them in accordance with their position in Γ, we obtain a block matrix $\Phi\in R_{\underline{m}\times\underline{n}}$, where $m_i:=q_\alpha$ if $i\in\mathcal{I}_\alpha$ and $n_j:=p_\beta$ if $j\in\mathcal{I}_\beta$. In the same way, the automorphisms $f\in\operatorname{Aut}_\Gamma P$ and $g\in\operatorname{Aut}_\Gamma Q$ are determined by nonsingular matrices from $\Gamma_{\underline{m}\times\underline{m}}$ and $\Gamma_{\underline{n}\times\underline{n}}$.

Hence, the problem of classifying modules over Γ reduces to the canonical form problem for matrices $\Phi\in R_{\underline{m}\times\underline{n}}$ up to transformations

$$\Phi\longmapsto F\Phi G,\qquad F\in\Gamma^*_{\underline{m}\times\underline{m}},\quad G\in\Gamma^*_{\underline{n}\times\underline{n}}. \tag{41}$$

Let

$$H_1,\dots,H_t \tag{42}$$

be the vertical strips of Φ with respect to $\underline{m}\times\underline{n}$ partition. The condition $\operatorname{Ker}\varphi\subset\operatorname{Rad}P$ from (37) means that

> there are not an equivalence class $\mathcal{I}_\alpha=\{j_1,\dots,j_{h(\alpha)}\}$ and a transformation (41) making zero the last column in each of $H_{j_1},\dots,H_{j_{h(\alpha)}}$ simultaneously. (43)

Step 2: an estimate. Let the module M in (36) has dimension at most d. By (36), (39), and the condition $\operatorname{Im}\varphi\subset\operatorname{Rad}Q$ from (37),

$$q_1+\dots+q_r=\dim Q/\operatorname{Rad}Q\leqslant\dim Q/\operatorname{Im}\varphi=\dim M\leqslant d. \tag{44}$$

Each summand $(e_\alpha\Gamma)^{p_\alpha}$ in the decomposition (39) of P determines the equivalence class $\mathcal{I}_\alpha=\{j_1,\dots,j_{h(\alpha)}\}$ and corresponds to the strips $H_{j_1},\dots,H_{j_{h(\alpha)}}$ of Φ (see (42)); these strips are reduced by simultaneous elementary transformations and each of them has p_α columns.

Let us prove that

$$p_\alpha\leqslant[q_1,\dots,q_r]\cdot[l_{1\alpha},\dots,l_{r\alpha}]^T, \tag{45}$$

where $[l_{1\alpha},\dots,l_{r\alpha}]^T$ is a column of the matrix (33). Put

$$n_\iota=[q_1,\dots,q_r]\cdot[\dim e_1\mathit{\Gamma} e_{j_\iota j_\iota},\dots,\dim e_r\mathit{\Gamma} e_{j_\iota j_\iota}]^T,\qquad 1\leqslant\iota\leqslant h(\alpha),$$

where e_{jj} are matrix units. By (32),

$$[q_1,\dots,q_r]\cdot[l_{1\alpha},\dots,l_{r\alpha}]^T=n_1+\dots+n_{h(\alpha)}.$$

Suppose that (45) does not hold, i.e.

$$p_\alpha\geqslant n_1+\dots+n_{h(\alpha)}+1,$$

and show that there is a transformation making zero the $(n_1+\dots+n_{h(\alpha)}+1)$st column in each of $H_{j_1},\dots,H_{j_{h(\alpha)}}$ simultaneously, to the contrary with (43). It suffices to show that there is a transformation making zero the $(n_1+\dots+n_{h(\alpha)}+1)$st column in all free blocks from $H_{j_1},\dots,H_{j_{h(\alpha)}}$ since the other blocks are their linear combinations.

The number of rows in free blocks of H_{j_1} is equal to n_1; by elementary transformations of columns, we maximize the rank of the first n_1 columns of these blocks, and then make zero the other their columns (by the definition of admissible transformations, the same transformations are produced within the strips $H_{j_2},\dots,H_{j_{h(\alpha)}}$). The number of rows in free blocks of H_{j_2} is n_2; by elementary transformations with the $n_1+1,n_1+2,\dots$ columns, we maximize the rank of the $n_1+1,n_1+2,\dots\ n_1+n_2$ columns of these blocks, and then make zero the $n_1+n_2+1,n_1+n_2+2,\dots$ columns in free blocks of H_{j_2} (the same transformations are produced within the strips $H_{j_1},H_{j_3}\dots,H_{j_{h(\alpha)}}$; they do not spoil the made zeros in H_{j_1}), and so on. At last, we reduce $H_{j_{h(\alpha)}}$ and obtain Φ in which the $(n_1+\dots+n_{h(\alpha)}+1)$st column is zero in all free boxes of $H_{j_1},\dots,H_{j_{h(\alpha)}}$. This proves (45).

Therefore, each module M of dimension at most d may be given by a sequence (36), in which P and Q are of the form (39) with p_i and q_j satisfying (44) and (45). To make

$$p_\alpha=[q_1,\dots,q_r]\cdot[l_{1\alpha},\dots,l_{r\alpha}]^T,$$

we add, if necessary, additional summands to the decomposition (39) of P and put φ equaling 0 on the new summands. Correspondingly, we omit the first condition in (37) and the condition (43) on the matrix Φ. The number of free entries in Φ becomes equal to

$$[q_1,\dots,q_r]L[p_1,\dots,p_r]^T=[q_1,\dots,q_r]L\cdot([q_1,\dots,q_r]L)^T;$$

this proves (34) in view of (44) and Theorem 3.2. $\square$

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